[mit N als Termdichte gemäß Gl. (27)] und hängt nur von $|\boldsymbol{q}| = q$ ab. Man führt jetzt Gl. (Z 3) in Gl. (Z 2) ein, löst dann Gl. (Z 2) nach $g_\omega(\boldsymbol{q}, \boldsymbol{k})$ auf [drückt also $g_\omega(\boldsymbol{q}, \boldsymbol{k})$ durch $K_\omega(q)$ aus] und integriert schließlich unter Beachtung von

$$\frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\vartheta}{2\,|\omega| + i\,v_\mathrm{F}\,q\cos\vartheta + v_\mathrm{F}/l} = \frac{\tan^{-1}\zeta_\omega\,q}{v_\mathrm{F}\,q} \qquad (\mathrm{Z}\,4)$$

mit

$$1/\zeta_\omega = 2\,|\omega|/v_\mathrm{F} + 1/l \qquad (\mathrm{Z}\,5)$$

über alle Richtungen des Vektors $\boldsymbol{k}$. Verwendet man noch einmal Gl. (Z 3), so erhält man eine lineare Gleichung für $K_\omega(q)$ mit der Lösung

$$K_\omega(q) = \frac{2\,\pi\,N}{v_\mathrm{F}}\left[\frac{q}{\tan^{-1}\zeta_\omega\,q} - \frac{1}{l}\right]^{-1}. \qquad (\mathrm{Z}\,6)$$

Das stimmt überein mit einem Ergebnis, das WERTHAMER [7] auf ganz anderem, quantenmechanischem Wege gewonnen hat.

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The Variation of the Adiabatic Invariant of the Harmonic Oscillator

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The variation of the adiabatic invariant μ of a harmonic oscillator or gyrating particle in a time varying homogeneous magnetic field is given as an absolutely converging series. This solution is used to discuss the behaviour of the adiabatic invariant for slow and fast as well as small and large variations of the oscillator strength.

General estimates are obtained for the dependence of the oscillator strength which is twice differentiable or which is analytic. For the latter case it is found that $\Delta\mu$ tends to zero at least as $\exp[-(2\,d-\varepsilon)/\alpha]$ for $\alpha \to 0$; $2\,d$ is the width of the analytic strip of $\omega(\alpha\,t)$; α characterizes the slowness of the variation and ε is any number $2\,d > \varepsilon > 0$.

The problem of the Adiabatic Invariant has a more than 50 years old history. It played a role in the old quantum theory in establishing the quantum conditions.

More recently, the magnetic moment — the magnic flux through the cross section of the orbit of a spiraling particle — plays a role in plasma physics and astrophysical considerations, as the VAN ALLEN belt. In a succession of papers [1] it has been proven that the adiabatic invariants are indeed invariant to all orders, where the expansion parameter is the "slowness" of the variation of the magnetic field.

Attempts have been made to compute the actual change of the magnetic moment for the case of a spiraling particle in a homogeneous, time varying magnetic field by HERTWECK and SCHLÜTER [2] and by CHANDRASEKHAR [3]. The results are approximate but asymptotically not quite correct as shown by BACKUS, LENARD, and KULSRUD [4].

In this communication the solution of the problem is given as an absolutely converging series and some asymptotic results are discussed.

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[2] F. HERTWECK and A. SCHLÜTER, Z. Naturforschg. **12 a**, 844 [1957].

[3] S. CHANDRASEKHAR, in Plasma and Magnetic Field, edited by R. K. M. Landshoff, Stanford University Press, 1958.

[4] G. BACKUS, A. LENARD, and R. KULSRUD, Z. Naturforschg. **15 a**, 1007 [1960].

Equations and Solution

The equation of motion of the harmonic oscillator is given by

$$\frac{d^2x}{dt^2} + \omega^2(t)\, x = 0\,. \tag{1}$$

If complex functions are admitted for x, (1) also describes a gyrating particle in a homogeneous time varying magnetic field [2].

The adiabatic invariant (or magnetic moment) is defined by

$$\mu = \frac{1}{2}\left(\frac{dx}{dt}\frac{d\bar{x}}{dt} + \omega^2(t)\, x\,\bar{x}\right)\frac{1}{\omega(t)} \tag{2}$$

($\bar{x}$ is the complex conjugate of x).

We are interested in the change of the adiabatic invariant $\Delta\mu = \mu(t_1) - \mu(t_0)$; $t_1 > t_0$ for $\omega > 0$. If μ depends on t, $\Delta\mu$ will also vary in time. In order to make $\Delta\mu$ time independant we consider the limit $t_1 \to +\infty$, $t_2 \to -\infty$, and allow only those variations of ω such that:

$$\lim_{t\to-\infty} \omega(t) = \omega_0 \quad \text{and} \quad \lim_{t\to+\infty} \omega(t) = \omega_1\,.$$

Furthermore also the first two derivatives of ω are to disappear in these limits.

Using Chandrasekhar's transformations [3]

$$d\tau = \omega(t)\, dt\,, \qquad W = \sqrt{\omega}\, x\,. \tag{3}$$

(1) can be transformed into

$$\ddot{W} + (1 - f(\tau))\, W = 0 \tag{4}$$

where

$$f(\tau) = \ddot{\beta}/\beta; \qquad \beta = \sqrt{\omega(\tau)}$$

and

$$d/d\tau = {}^{\bullet}\,. \tag{5}$$

(4) is of the same form as (1) but contrary to (1), it has the same asymptotic solutions for $t \to \pm\infty$, because in these limits $f(\tau) = 0$.

We obtain from (2)

$$\mu = \frac{1}{2}\left[W\overline{W} + \left(\dot{W} + \frac{1}{2}\frac{\dot{\omega}}{\omega}W\right)\left(\dot{\overline{W}} + \frac{1}{2}\frac{\dot{\omega}}{\omega}\overline{W}\right)\right]. \tag{6}$$

The change in μ is easily calculated by multiplying (4) with $\dot{\overline{W}}$, adding the conjugate complex and integrating over time:

$$\Delta\mu = \frac{1}{2}\Delta(W\overline{W} + \dot{W}\dot{\overline{W}}) = \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\, f(\tau)\, \frac{d}{d\tau}(W\overline{W})\,; \tag{7}$$

Chandrasekhar solves (4) by an iteration procedure and stops at second order. We do essentially the same, but in writing down the whole series we are able to prove its absolute convergence. This property plays a mayor role in the following.

Let's first convert (4) into a Volterra Integral Equation by making use of the Green function of the oscillator equation:

$$W(\tau) = W_0(\tau) + \int_{-\infty}^{\tau} d\tau'\, \sin(\tau - \tau')\, f(\tau')\, W(\tau')\,. \tag{8}$$

Here $W_0(\tau)$ is the solution for $\tau \to -\infty$ which can be written as

$$W_0(\tau) = A\, e^{i\tau} + B\, e^{-i\tau}$$

or

$$W_0(\tau) = C\cos\tau + D\sin\tau\,. \tag{9}$$

The constants A, B and C, D are conveniently normalized to

$$|A|^2 + |B|^2 = 1 \quad \text{and} \quad |C|^2 + |D|^2 = 1\,. \tag{10}$$

Anticipating that the theory of Volterra equations applies [5] the solution of (8) may be written as

$$W(\tau) = W_0(\tau) - \int_{-\infty}^{\tau} H(\tau, \tau')\, W_0(\tau')\, d\tau'. \tag{11}$$

The resolvent kernel $H(\tau, \tau')$ statisfies the integral equations

$$K(\tau,\tau') + H(\tau,\tau') = \int_{\tau'}^{\tau} d\tau''\, K(\tau,\tau'')\, H(\tau'',\tau')$$

$$= \int_{\tau'}^{\tau} H(\tau,\tau'')\, K(\tau'',\tau')\, d\tau'' \tag{12}$$

where

$$K(\tau,\tau') = \sin(\tau - \tau')\, f(\tau') \tag{13}$$

is the kernel of the original equation (8).

The resolvent kernel may also be written as a series:

$$H(\tau,\tau') = \sum_0^{\infty} K_{n+1}(\tau,\tau') \tag{14}$$

where

$$K_{n+1}(\tau,\tau') = \int_{\tau'}^{\tau} d\tau''\, K(\tau,\tau'')\, K_n(\tau'',\tau)\,. \tag{15}$$

Thus the solution of (4) can be written as

$$W(\tau) = W_0(\tau) + \int_{-\infty}^{\tau} d\tau_1\, K(\tau,\tau_1)\, W_0(\tau_1) + \int_{-\infty}^{\tau} d\tau_1\, K(\tau,\tau_1) \int_{-\infty}^{\tau_1} d\tau_2\, K(\tau_1,\tau_2)\, W_0(\tau_2) + \dots \tag{16}$$

The above solution is only sensible, if the series (14) and (16) converge. The usual theory of

[5] e. g. F. G. Tricomi, Integral Equations, Interscience Publishers, New York 1957.

VOLTERRA equations cannot be applied because it is based upon the assumption that $K(\tau, \tau')$ is a L_2-kernel for both variables. In our case it is immediately seen that

$$\int_{-\infty}^{+\infty} d\tau\, K^2(\tau, \tau')$$

does not exist. The situation can be salvaged however, if we place $f(\tau)$ into the class L_1 i. e. if

$$\int_{-\infty}^{+\infty} d\tau\, |f(\tau)| = C < \infty\,. \qquad (17)$$

We prove the convergence of (12) by complete induction and state that

$$|K_{n+1}(\tau, \tau')| \leqq |f(\tau')| \frac{1}{n!} \Big[\int_{\tau'}^{\tau} d\tau''\, |f(\tau'')|\Big]^n. \qquad (18)$$

It is certainly true for $n=0$. To prove it for $N+2$ if it is assumed to be true for $n=N+1$ we write:

$$|K_{N+2}(\tau, \tau')| \leqq \Big|\int_{\tau'}^{\tau} dz\, K(\tau, z)\, K_{N+1}(z, \tau')\Big|$$

$$\leqq \int_{\tau'}^{\tau} dz\, |f(z)|\, |f(\tau')| \frac{1}{N!} \Big[\int_{\tau'}^{z} dz'\, |f(z)|\Big]^N$$

$$= |f(\tau')| \int_{\tau'}^{\tau} \frac{\partial}{\partial z} \frac{1}{(N+1)!} \Big[\int_{\tau'}^{z} dz'\, |f(z')|\Big]^{N+1}$$

$$= |f(\tau')| \frac{1}{(N+1)!} \Big[\int_{\tau'}^{\tau} dz\, |f(z)|\Big]^N\,.$$

Thus $|H(\tau, \tau')|$ has the majorant

$$|f(\tau')| \exp\{\int_{\tau'}^{\tau} dz\, |f(z)|\} \qquad (19)$$

and all expressions in (11), (12), (14) and (16) exist.

Representation of the Solving Kernel $H(\tau, \tau')$ by the Solutions of (4)

A representation of $H(\tau, \tau')$ in terms of the solutions of (4) can be formed. We differentiate the left hand side and the middle part of (12) twice with respect to τ. The result is, after some obvious substitutions

$$\frac{\partial^2}{\partial \tau^2} H(\tau, \tau') + [1 - f(\tau)]\, H(\tau, \tau') = 0.$$

Thus if $W_1(\tau)$ and $W_2(\tau)$ are two independent solutions of (4), $H(\tau, \tau')$ may be written as

$$W_1(\tau)\, g_1(\tau') + W_2(\tau)\, g_2(\tau'), \qquad (20)$$

g_1 and g_2 being arbitrary functions.

Differentiation of the right and left hand side of (12) with respect to τ', after a division by $f(\tau')$ results in

$$\frac{\partial^2}{\partial \tau'^2} \Big[\frac{H(\tau, \tau')}{f(\tau')}\Big] + (1 - f(\tau')) \frac{H(\tau, \tau')}{f(\tau')} = 0\,.$$

Obviously $H(\tau, \tau')$ must also be of the form

$$f(\tau')\,[W_1(\tau')\, h_1(\tau) + W_2(\tau')\, h_2(\tau)]\,. \qquad (21)$$

In order to reconcile (20) and (21), $H(\tau, \tau')$ must be written as

$$f(\tau')\{W_1(\tau)\,[a_1 W_1(\tau') + a_2 W_2(\tau')] + W_2(\tau)\,[b_1 W_1(\tau') + b_2 W_2(\tau')]\}\,, \qquad (22)$$

where a_1, a_2, b_1, b_2 are constants. To determine those insert (22) into (12) and assume that for $\tau \to -\infty$

$$W(\tau) = W_1(\tau) \to e^{i\tau}; \quad \overline{W(\tau)} = W_2(\tau) \to e^{-i\tau}. \qquad (23)$$

In this limit the right hand member of (12) vanishes to higher order than the left hand side and the constants are determined to

$$a_1 = b_2 = 0; \; a_2 = -1/2\,i\,; \; b_2 = +1/2\,i$$

and we thus obtain:

$$H(\tau, \tau') = -\frac{1}{2\,i} f(\tau')\,[W(\tau)\,\overline{W(\tau')} - \overline{W(\tau)}\, W(\tau')]\,. \qquad (24)$$

If we choose

$$W_1(\tau) \to \sin\tau, \quad W_2(\tau) \to \cos\tau, \quad \tau \to -\infty \qquad (25)$$

we obtain in the same manner:

$$H(\tau, \tau') = -f(\tau')\,[W_1(\tau)\, W_2(\tau') - W_2(\tau)\, W_1(\tau')]. \qquad (26)$$

We shall use this formula in one of the following estimates.

Definition of Slow and Fast Transitions

We want to study the variation of μ if $f(\tau)$ changes very slowly compared to a gyration period of the particle. For this purpose we write formally $\omega(\alpha t)$ instead of $\omega(t)$ in (1) and (2). (3) is replaced by

$$\alpha\,\tau = \int_0^{\alpha\tau} \omega(\alpha\, t)\, d(\alpha\, t)\,.$$

Then $f(\tau)$ is replaced by

$$\alpha^2 f(\alpha\,\tau) \qquad (27)$$

in (5) and (4).

We now define a "slow transition" by the limit $\alpha \to 0$. The opposite, a rapid change of $\omega(t)$ has already been treated by HERTWECK and SCHLÜTER and our method is not appropriate to treat the same case. However, if we generalize (27) into

$$\alpha\,\lambda(\alpha)\,f(\alpha\,\tau) \tag{28}$$

where λ is an appropriately chosen amplitude factor, which may depend on α, we are able to treat the limit $\alpha \to \infty$ of "fast transitions". A class of $\omega(t)$ is found for which a rigourously constant adiabatic invariant can be defined.

Fast Transitions

We consider for simplicity a function $f(\alpha\,\tau)$ which is zero outside $0 \leqq \alpha\,\tau \leqq T$. It is easy to generalize the consideration for the case where $f(\alpha\,\tau)$ becomes not zero but only very small outside this interval.

If we write (16) as

$$W(\tau) = W_0(\tau) + W_1(\alpha,\tau) + W_2(\alpha,\tau) + \ldots \tag{29}$$

then for large α $|W_n(\alpha,\tau)|$ can be majorized by

$$|W_n(\alpha,\tau)| \leqq \begin{cases} 0; & \tau \leqq 0 \\ 2^{n-1}(\lambda(\alpha)\,I)^n (T/\alpha)^{n-1} \cdot (|\sin\tau| + (T/\alpha)|\cos\tau|); & \tau > 0;\ n \geqq 1 \end{cases} \tag{30}$$

where $$I = \int_{-\infty}^{+\infty} |f(\alpha\,\tau)|\,\mathrm{d}(\alpha\,\tau)\,.$$

(30) is easily shown to hold for $n=1$ and proved generally by induction. The series (29) certainly converges if we choose $\lambda(\alpha) = \text{const.}$ and make α large enough.

In the limit $\alpha \to \infty$ ist is seen from (28) that

$$\lambda\,\alpha\,f(\alpha\,\tau) \to \lambda\,\delta(\tau) \tag{31}$$

and (30) shows that the solution is given by the first 2 terms of (29):

$$W(\tau) = W_0(\tau) + \lambda(\sin\tau)\,W_0(0)\,. \tag{32}$$

This solution can be interpreted in physical terms by putting

$$\omega(t) = 1 - \lambda\,\delta(t)\,. \tag{33}$$

This means that the string of the pendulum experiences a sudden elongation and immediate contraction to the original length or vice versa. On the other hand we may also interpret $f(\tau)$ as $\ddot{\beta}/\beta$ as in (5). After integration we obtain

$$\begin{aligned} \tau &= \omega_0\,t \quad ; \quad \omega = \omega_0 > 0 \quad ; \quad t \leqq 0 \\ \tau &= \frac{\omega_0\,t}{1-\lambda\,\omega_0\,t}\,; \quad \omega = \omega_0(1+\lambda\,\tau)^2 \\ &\qquad\qquad = \omega_0/(1-\omega_0\,\lambda\,t)^2\,; \quad t \geqq 0\,. \end{aligned} \tag{34}$$

Thus (31) also represents a pendulum, the length l of which is contracted or elongated according to

$$l = l_0(1-\lambda\,\omega_0\,t)^4 \quad \text{where} \quad \omega_0 = \sqrt{g/l_0}\,. \tag{35}$$

Note that $\frac{1}{2}\,(W\overline{W} + \dot{W}\dot{\overline{W}})$ is an exact invariant for all times except $t=0$, whereas μ is not because $\mathrm{d}\omega/\mathrm{d}t$ does not disappear for $t>0$.

From (9), (10) and (32) it is easily calculated that

$$\begin{aligned} &\Delta \tfrac{1}{2}\,(W\overline{W} + \dot{W}\dot{\overline{W}}) \\ &\quad = i\,\lambda(A\overline{B} - \overline{A}B) + \tfrac{1}{2}\lambda^2(1 + A\overline{B} + \overline{A}B)\,. \end{aligned} \tag{36}$$

If $W(\tau)$ is real, then A and B contain one arbitrary phase factor, however 2, if $W(\tau)$ is complex. In the first case we must therefore write

$$A = |A|\,e^{i\varphi_0}; \qquad B = |B|\,e^{-i\varphi_0};$$

and for the second case

$$A = |A|\,e^{i\varphi_1}; \qquad B = |B|\,e^{i\varphi_2}.$$

If the initial phases φ_0, or φ_1 and φ_2 are averaged, all terms bilinear in A and B in (36) vanish and we obtain

$$\Delta \tfrac{1}{2}(W\overline{W} + \dot{W}\dot{\overline{W}}) = \tfrac{1}{2}\lambda^2\,. \tag{37}$$

General Estimate

The assumption (17) can also be written

$$\frac{1}{2}\int_{-\infty}^{+\infty} \left| \frac{\mathrm{d}^2\omega(t)}{\mathrm{d}t^2} - \frac{3}{2}\,\frac{1}{\omega(t)}\left(\frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\right)^2 \right| \frac{\mathrm{d}t}{\omega(t)^2} < \infty\,. \tag{38}$$

Therefore $\omega(t)$ must be greater than zero, at least piecewise differentiable and its derivative must vanish faster than

$$O(t^{-1/2}) \quad \text{for} \quad t \to \pm\infty\,.$$

We will derive some general estimates.

We define $W_1(\tau)$ and $W_2(\tau)$ as in (25) and write down (11) for W_1 and W_2, making use of (26):

$$W_1(\tau) = \sin\tau + W_1(\tau) \int_{-\infty}^{\tau} f(\tau')\, W_2(\tau') \sin\tau'\, d\tau' - W_2(\tau) \int_{-\infty}^{\tau} f(\tau')\, W_1(\tau') \sin\tau'\, d\tau';$$

$$W_2(\tau) = \cos\tau + W_1(\tau) \int_{-\infty}^{\tau} f(\tau')\, W_2(\tau') \cos\tau'\, d\tau' - W_2(\tau) \int_{-\infty}^{\tau} f(\tau')\, W_1(\tau') \cos\tau'\, d\tau'. \quad (39)$$

If we define

$$\begin{Bmatrix} G_{1s}(\tau), G_{2s}(\tau) \\ G_{1c}(\tau), G_{2c}(\tau) \end{Bmatrix} = \int_{-\infty}^{\tau} f(\tau') \begin{Bmatrix} W_1(\tau') \sin\tau', & W_2(\tau') \sin\tau' \\ W_1(\tau') \cos\tau', & W_2(\tau') \cos\tau' \end{Bmatrix} d\tau' \quad (40)$$

$$G = \mathrm{Max}\, | G_{\mu,\nu} | \quad (41)$$

and $$D = (1 - G_{2s})(1 + G_{1c}) + G_{1s} G_{2c} \quad (42)$$

one can write (39) as

$$W_1(\tau) = (1/D)\,[(1 + G_{1c}) \sin\tau - G_{1s} \cos\tau];$$
$$W_2(\tau) = (1/D)\,[G_{2c} \sin\tau + (1 - G_{2s}) \cos\tau]. \quad (43)$$

If $G \ll 1$ one obtains to first order in the G's

$$W_1(\tau) = (1 + G_{2s}) \sin\tau - G_{1s} \cos\tau;$$
$$W_2(\tau) = G_{2s} \sin\tau + (1 - G_{1c}) \cos\tau. \quad (44)$$

In order to estimate G, (19) is introduced into (11) and the following inequality results:

$$| W(\tau) | \leqq W_0(\hat{\tau})_{\max} \exp\{ \int_{-\infty}^{\tau} | f(z) |\, dz \}. \quad (45)$$

For $\hat{\tau} = \tau$ $| W_0(\hat{\tau}) |$ assumes its maximum value.

With (45) one obtains from (40)

$$G \leqq \exp[\int_{-\infty}^{+\infty} | f(z) |\, dz] - 1. \quad (46)$$

G becomes very small if we let go $\alpha \to 0$ in (27) or $\lambda \to 0$ in (28). We thus see that for slow changes (small amplitudes) the "perturbation" introduced by the change of the magnetic field goes to zero at least as

$$\alpha \int_{-\infty}^{+\infty} | f(\tau) |\, d\tau \quad \text{or} \quad \lambda \int_{-\infty}^{+\infty} | f(\tau) |\, d\tau \quad \text{respectively.}$$

From the general solution of (4), $C W_1(\tau) + D W_2(\tau)$ we obtain

$$\mu(\tau \to -\infty) = 1 \quad \text{and}$$
$$\mu(\tau \to +\infty) = | C(1 + G_{2s}(\infty)) + D\, G_{2c}(\infty) |^2 + | -C\, G_{1s}(\infty) + D(1 - G_{1c}(\infty)) |^2.$$

Thus the change of the adiabatic invariant can be estimated to be:

$$| \Delta\mu | \lesssim 2\, G (1 + G)\, [1 + 2\, \mathrm{Re}(\overline{C}\, D)]. \quad (47)$$

Holomorphic $\omega(t)$

The degree of constancy of μ can be very much improved if more stringent conditions are imposed on $\omega(t)$. We assume now that $\omega(\alpha t)$ is holomorphic in a strip around the real axis of width $2D$. Naturally this strip is symmetric with respect to the real axis because $\omega(t)$ is a real function. If the general solution (9.1) is inserted into (7), we obtain because of (10)

$$\Delta\mu = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau\, f(\tau)\, \frac{d}{d\tau} [W(\tau)\, \overline{W(\tau)} + (A\, B\, W(\tau)^2 + \text{c.c.})]. \quad (48)$$

From (16) it is easily seen that $W(\tau)$ may be written as

$$W(\tau) = e^{i\tau}\, G(\tau) \quad (49)$$

with

$$G(\tau) = 1 + \int_{-\infty}^{\tau} d\tau_1 \sin(\tau - \tau_1)\, e^{i(\tau_1 - \tau)}\, f(\tau_1)\, d\tau_1 + \int_{-\infty}^{\tau} d\tau_1 \sin(\tau - \tau_1)\, e^{i(\tau_1 - \tau)}\, f(\tau_1) \int_{-\infty}^{\tau} \sin(\tau_1 - \tau_2)\, e^{i(\tau_2 - \tau_1)}\, f(\tau_2)\, d\tau_2 + \dots . \quad (50)$$

Note that

$$\frac{dG}{d\tau} \int_{-\infty}^{\tau} d\tau'\, e^{-2i(\tau - \tau')}\, f(\tau')\, G(\tau') \quad (51)$$

performing the differentiations at the first term in (48) gives

$$\tfrac{1}{2} \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{\tau} d\tau'\, f(\tau)\, f(\tau')\, [e^{-2i(\tau - \tau')}\, G(\tau')\, \overline{G(\tau)} + e^{+2i(\tau - \tau')}\, G(\tau)\, \overline{G(\tau')}] \quad (52)$$

and the second term
$$A\bar{B}\int_{-\infty}^{+\infty} d\tau\, f(\tau)\, e^{2i\tau}\, G(\tau)\left[i\, G(\tau)+\frac{dG(\tau)}{d\tau}\right]+\text{c. c.} \tag{53}$$

We observe that in (52) upon an interchange of τ and τ' the first term in parenthesis goes over into the second and vice versa. Thus the integrand is symmetric in τ and τ'. We may therefore extend the integration in the (τ,τ')-plane from the aerea above the diagonal $\tau=\tau'$ over the total plane and write

$$\tfrac{1}{2}\left|\int_{-\infty}^{+\infty} d\tau\, f(\tau)\, e^{2i\tau}\, G(\tau)\right|^2. \tag{54}$$

Let us now introduce α by (27), substitute $u_\nu=\alpha\,\tau_\nu$ in (50) and consider small α. Then (50) can be written:

$$G(\alpha,u)=1+\alpha\int_{-\infty}^{u} du_1 \sin(u-u_1/\alpha)\, e^{i(u_1-u)/\alpha} f(u_1)$$
$$+\alpha^2\int_{-\infty}^{u} du_1 \sin(u-u_1/\alpha)\, e^{i(u_1-u)/\alpha} f(u_1)\int_{-\infty}^{u_1} du_2 \sin(u_1-u_2/\alpha)\, e^{i(u_2-u_1)/\alpha} f(u_2)+\ldots \tag{55}$$

If the coefficients of (4) are non zero and analytic in a certain aerea then the solutions of (4) are analytic in the same aerea. As $G(\tau)$ is related to $W(\tau)$ by (49) it is also an analytic function in the strip around the u-axis of width $2D$. We may therefore displace the path of integration in all the integrals which occur in G by the substitution $\alpha\,\tau_\nu=i\,d+s_\nu$, $d<D$ and obtain:

$$G(\alpha,i\,d+s)=1+\alpha\int_{-\infty}^{s} ds_1 \sin(s-s_1/\alpha)\, e^{i(s_1-s)/\alpha} f(i\,d+s) \tag{56}$$
$$+\alpha^2\int_{-\infty}^{s} ds_1 \sin(s-s_1/\alpha)\, e^{i(s_1-s)/\alpha} f(i\,d+s_1)\int_{-\infty}^{s_1} ds_2 \sin(s_1-s_2/\alpha)\, e^{i(s_2-s_1)/\alpha} f(i\,d+s_2)+\ldots$$

The integral $\alpha^2\int_{-\infty}^{+\infty} d\tau\, f(\alpha\,\tau)\, e^{2i\tau}\, G(\alpha,\tau)$ occuring in (54) can be written

$$\alpha\int_{-\infty}^{+\infty} ds\, f(i\,d+s)\, e^{-2d/\alpha+2is/\alpha}\, G(\alpha,i\,d+s). \tag{57}$$

An upper bound of this integral is given by

$$\alpha\, e^{-2d/\alpha}\int_{-\infty}^{+\infty} ds\,|f(i\,d+s)|\,\{1+\alpha\int_{s_1}^{s} ds_1\,|f(i\,d+s_1)|+\alpha^2\int_{-\infty}^{s} ds_1\,|f(i\,d+s_1)|\int_{-\infty}^{s_1} ds_2\,|f(i\,d+s_2)| \tag{58}$$
$$+\alpha^3\int_{-\infty}^{s} ds_1\,|f(i\,d+s_1)|\int_{-\infty}^{s_1} ds_2\,|f(i\,d+s_2)|\int_{-\infty}^{s_2} ds_3\,|f(i\,d+s_3)|+\ldots\}.$$

This kind of series has already been treated and can be summed up to an exponential to give

$$\exp\{-2\,d/\alpha\}\cdot\exp\{\alpha\int_{-\infty}^{+\infty} ds\,|f(i\,d+s)| \tag{59}$$

If we consider the change of μ, averaged over the initial phases the term (53) drops out and we obtain:

$$\Delta\mu<\frac{1}{2}\exp\left\{-\frac{4\,D-\varepsilon}{\alpha}\right\}$$
$$\cdot\exp\{2\,\alpha\int_{-\infty}^{+\infty} ds\,|f(i(D-\varepsilon)+s)|; \tag{60}$$

ε is any number $0<\varepsilon<D$.

We have thus shown that for analytic $\omega(t)$ the variation of the magnetic moment indeed goes to zero as $\exp[-2\delta/\alpha]$ as supposed by KULSRUD and the constant δ is equal to the width of the analytic strip. The estimate (60) is still pessimistic because we have neglected all phase relations which occur in (56). Thus $\int_{-\infty}^{+\infty} ds\,|f(i\,d+s)|$ diverges as d approches D if f has a pole. One would obtain a still lower limit of $\Delta\mu$ if the functional behavior of $f(i\,d+s)$ in the neighborhood of the singularity is properly taken into account.

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It has been brought to the attention of the authors only recently that A. CAVALLIERE, B. CROSIGNANI, and F. GRATTON (Nuovo Cimento **33**, 1338 [1964]) have treated the same problem by similar methods. They use the convergent series (16), but assume that $f(\tau)$ be bounded whereas we assume $f(\tau)$ to belong to the class L_1. For holomorphic $\omega(t)$ CAVALLIERE et al. arrive at the same conclusions as we do. In the other aspects of the treatment there is no overlapping.